Probing Students' Numerical Misconceptions in School Algebra

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The study was designed to probe students' thinking about which numerical values can be assigned to algebraic letters. The data from students in Year 7 (n=533), Year 8 (n=377) and Year 9 (n=172) was analysed using response patterns. The data confirmed that each year contained students with two misconceptions; *Different Letter means Different Number* and the *Empty Box* misconceptions. The findings provide support for the Steinle et al. (2009) hypothesis that a previously identified response pattern is a subset of the *Empty Box* misconception.

There has been considerable research into students' difficulties with algebra. Students who have an incomplete or incorrect interpretation of the meaning of letters, or of the equals sign, will have difficulty making sense of this important topic in mathematics. For example, in algebra, letters are used for both unknown (specific) numbers or as generalised numbers (i.e. variables). Christou and Vosniadou (2012) noted that, not only do students have a tendency to substitute only specific numbers for letters, they often limit the specific numbers to natural numbers.

As well as difficulties interpreting the letters used in algebra, students can also have a limited interpretation of the equals sign as an instruction to 'do' something (i.e operational), rather than as indicating balance (i.e. relational). Asquith, Stephens, Knuth and Alibali (2007) conducted research with a small sample of middle school teachers in the USA and concluded that these teachers overestimated their students' ability to give a relational definition of the equals sign.

Lins & Kaput (2004) suggest that the tradition of *arithmetic then algebra* has contributed to students' difficulties and recommend an early introduction to *algebraic reasoning* (but not to literal symbols). Stephens (2008) investigates student's transition from arithmetic to algebra and the usefulness of students developing relational thinking with number sentences to assist with this transition to literal symbols.

Many researchers have investigated students' understanding about the use of letters in algebra; e.g. MacGregor & Stacey (1997). In contrast to careless errors, misconceptions (also referred to as 'synthetic conceptions', by Christou & Vosniadou, 2012) lead to predictable errors in student work. Fujii (2003) categorised students into four groups based on their pattern of responses to two items and interviews. Of these four groups, two have particular misconceptions and are the focus of this paper.

Steinle, Gvozdenko, Price, Stacey and Pierce (2009) used the term 'numerical misconceptions' to distinguish these misconceptions (involving which numerical values can be assigned to letters) from 'non-numerical misconceptions' (such as letter as object). They modified items in Fujii (2003) with slightly different instructions in order to use in an online environment (www.smartvic.com). Specific Mathematics Assessments that Reveal Thinking (smart-tests) are designed for teachers to use for diagnostic purposes and for planning future teaching. These electronic tests are quick for students to complete (5 -10 minutes) and the teacher is instantaneously provided with an online diagnosis for each student. Accompanying the diagnoses are explanations of the associated likely student thinking and/or reasons for the errors, along with teaching suggestions and links to other resources. These suggestions are designed to increase teachers' mathematical pedagogical

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content knowledge for specific topics, so affect both short-term teaching (of current students) and long-term teaching (of future students).

Steinle et al. (2009) confirmed the students' misconceptions highlighted by Fujii (2003) about the numerical values of letters in algebra and found some new response patterns which suggest possible new misconceptions. The current study has been designed to test their hypothesis that the modification of the Fujii (2003) items for use in an online environment has caused a split within the group of students who have a particular misconception we call Empty Box. Students with this Empty Box (EB) misconception effectively treat the letters in an algebraic equation as if they are boxes, so $x + x + x = 12$ is interpreted as $\square + \square + \square = 12$, in which case, they believe that there are many correct solutions (rather than one), as each box can represent a different value.

Literature

Algebra is considered a difficult part of school mathematics mostly due to the need for students to develop appropriate interpretations of the abstract symbols involved. Kieran (2006) noted that "many conceptual adjustments are required of the beginning algebra student as these signs and symbols shift in meaning from those commonly held in arithmetic" (p. 13).

Christou and Vosniadou (2012) note that if the new information that a student is presented with is incompatible with what is already known, it creates either "fragmentation or systematic misconceptions –otherwise known as synthetic conceptions" (p. 5). Various misconceptions are well-known; for example, in the number topic, many students believe *multiplication makes bigger and division makes smaller* (see for example, Lim, 2011). Various misconceptions within algebra have been reported; for example, the *letter as object misconception* (see Küchemann, 1981; MacGregor & Stacey, 1997; Warren, 1998 and Akhtar & Steinle, submitted) as well as *different letters stand for different numbers* and *the same letter does not necessarily stand for the same number* (see Fujii, 2003; Steinle et al. 2009).

Küchemann (1981) identified that students believed that different letters stand for different numbers. For example, over 50% of students (11 to 16 years of age) chose the option "never true" when given this equation: L + M + N = L + P + N. The same thinking was verified by Booth (1984) who used the same item in a study with 13 to 15 year olds. Again, just over 50% chose "never true".

Fujii (2003) studied students' difficulties with two items involving appropriate choices for the numerical value of letters. Initially he analysed students' responses to each item separately but then he realized that analysing responses across both items was more revealing. Based on additional evidence from interviews, he grouped students into four groups; Both problems are correct, Type A, Type B and Others. He defined Type A as "different letters stand for different numbers" and Type B as "the same letter does not necessarily stand for the same number" (p. 52).

Interestingly, Fujii (2003) studied Japanese and American students' responses and noted the samples showed a similar tendency; just over 10% of students in 8^{th} Grade (for example) were correct on both items.

Steinle et al. (2009) built on Fujii's work, modifying the items slightly for an online environment; see Figure 1. Students were provided with a drop-down box so that they could choose the option Right or Wrong on each of the six tasks.

Item 498	Item 502
Some students had to find some values of x to make this equation true: $x + x + x = 12$ Mark the work of each student. Mary wrote $x = 2$, $x = 5$ and $x = 5$ Millie wrote $x = 9$, $x = 2$ and $x = 1$ Mandy wrote $x = 4$	Some students had to find some values of x and y to make this equation true: $x + y = 16$ Mark the work of each student. John wrote $x = 6$, $y = 10$ Jack wrote $x = 8$ and $y = 8$ James wrote $x = 9$ and $y = 7$

Figure 1. Original items used in Steinle et al. (2009).

Similar to Fujii (2003), a student's pattern of responses to the full set of six tasks were used to classify students into groups. Steinle et al. (2009) confirmed the same misconceptions found by Fujii and used the following labels: Expert (Both problems are correct), Different Letter means Different Number (Type A), Empty Box (Type B) and Unclassified (Others).

Table 1 provides the response patterns and accuracy patterns for four groups of students. For example, if a student responded to these 6 tasks with this response pattern: *Wrong, Wrong, Right, Right, Right, Right*, then the student was labelled Expert. Table 1 contains this Expert response pattern and the Expert accuracy pattern (√,√,√,√,√,√), as well as number of students who responded in this way (n=23). The final row of the table indicates that this pattern occurred four times more than would be predicted if students were choosing their answers randomly.

Table 1
Response Patterns from Steinle et al. (2009)

Abbreviated tasks within Item 498 and Item 502	Expert		DLDN		EB		New1	
	Resp Patt	Acc Patt	Resp Patt	Acc Patt	Resp Patt	Acc Patt	Resp Patt	Acc Patt
Mary x = 2, x = 5, x = 5	W	√	W	√	R	×	R	×
Millie x = 9, x = 2, x = 1	W	√	W	√	R	×	R	×
Mandy x = 4	R	√	R	√	R	√	W	×
John x = 6, y = 10	R	√	R	√	R	√	R	√
Jack x = 8, y = 8	R	√	W	×	R	√	R	√
James x = 9, y = 7	R	√	R	√	R	√	R	√
Frequency	23		70		74		29	
Ratio*	4.0		12.2		12.9		5.1	

Note: Accuracy Pattern (Acc Patt) indicates when a response of R (Right) or W (Wrong) is correct (√).
* The ratio of observed frequency to expected frequency if all students chose randomly.

The misconceptions Different Letter means Different Number (DLDN) and Empty Box (EB) occurred more than 12 times as often as would be expected if students' choices were random. The pattern labelled New1 was not predicted, and it occurred more often than the Expert pattern. Steinle et al. hypothesised that the modification of Fujii's item might have resulted in the splitting of the EB group on the Mandy task (see row 3 of Table 1). If students believe that we need any three numbers which add to 12 in Item 498, then $x=4$ (while correct from an Expert's point of view) is not a suitable answer, and these students would then choose *Wrong* (W) on the Mandy task.

The focus of this paper is to try to determine whether this hypothesis is reasonable. A new version of the instrument was created by including an additional task and the data from a new sample of students was analysed.

Methodology

The sample of the study was 1082 students from 20 secondary schools in Melbourne; Year 7 (n = 533), Year 8 (n = 377) and Year 9 (n = 172). The instrument was a test which consisted of two items, from the smart-test algebra module. The test was administered to the students by their teachers during their classes in the period January to October 2011. As the smart-test system is designed to be used by teachers for formative assessment, the results of Year 7 students should be interpreted with caution. If teachers are using these tests to determine what students know before they teach the topic, then results for Year 7 could be considered as pre-teaching or baseline data.

The research instrument (see Figure 2) was created by adding one new task to the first item; "Molly wrote $x = 4$, $x = 4$, $x = 4$", making a total of seven tasks in these two items.

Item 2254 (Variation of 498)	*Item 2269 (Same as 502)*
Some students had to find some values of x to make this equation true: $x + x + x = 12$ Mark the work of each student.	Some students had to find some values of x and y to make this equation true: $x + y = 16$ Mark the work of each student.
Mary wrote $x = 2$, $x = 5$, $x = 5$ Millie wrote $x = 9$, $x = 2$, $x = 1$ Mandy wrote $x = 4$ Molly wrote $x = 4$, $x = 4$ and $x = 4$	John wrote $x = 6$, $y = 10$ Jack wrote $x = 8$, $y = 8$ James wrote $x = 9$ and $y = 7$

Figure 2. Items from updated version of smart-test.

Results and Discussion

Responses to six original tasks

In this section, the results for the new sample (n = 1082) on the six original tasks will be compared with the previous study.

As before, a *pattern recognition script* was used to search the data for common response patterns to the original six tasks. The top four patterns are the same as found previously; these are listed in decreasing order in Table 2. As before, DLDN, EB and New1 all occur more often than the Expert pattern, and the last row of the table indicates that these patterns occur between 6 and 14 times more often than would be predicted if students were choosing at random. In this study, New1 occurs more often than EB. These patterns account for 692 students (64%) of the total sample (n =1082).

Hence, we have confirmation that these response patterns do appear in the data and Fujii's interviews provide the evidence that students do have the two particular misconceptions he described, and that we labelled DLDN and EB.

The high frequency of New1 provides evidence that it was not an anomaly in the previous data. Whether this group of students has a new misconception or is part of the EB group who have responded differently to the Mandy task is to be determined in the following section.

Table 2
Frequency of the most common patterns to 6 tasks

Patterns	DLDN	New1	EB	Expert
Frequency	232	187	157	116
Ratio*	13.7	11.1	9.3	6.9

* The ratio of observed frequency to expected frequency if all students chose randomly.

Responses to Molly task

Steinle et al. (2009) predicted that modifications to Fujii's items for the online test had led to a split of the students with EB thinking on this task: Mandy x=4. Hence, the extra task (Molly $x = 4$, $x = 4$, $x = 4$) was introduced in item 2254 to study the behaviour of the students who had the response patterns EB and New1.

The percentage of each group of students who chose Right and Wrong on the Molly task was determined and is shown in Figure 3. Just over 20% of the Experts indicated Molly was wrong, while very few of the students in EB and New1 (3% and 5%, respectively) chose this response. The similar behaviour of the EB and New1 groups on the Molly task, and the fact that these differ from both DLDN and Expert, provides support for the hypothesis that EB and New1 students have very similar thinking, and that the different format of the Mandy task has split one group into two.

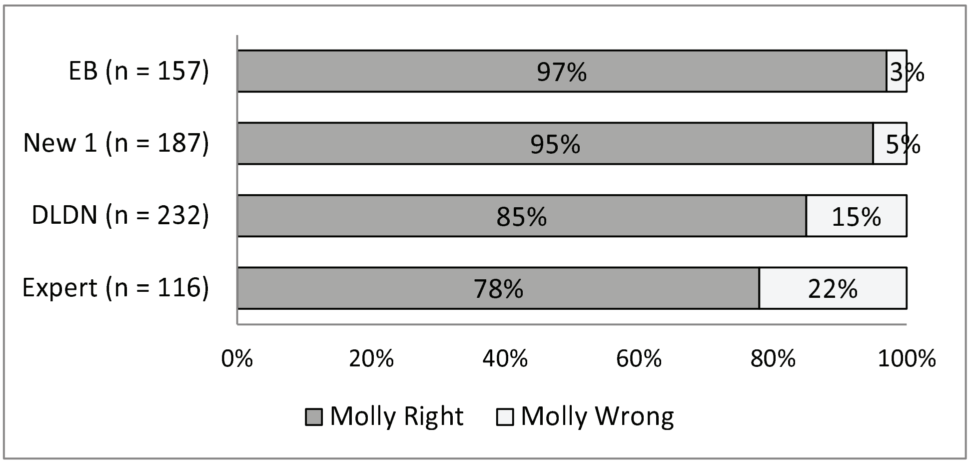

Figure 3. Students' response on Molly task.

Analysis by year level

The sample was comprised of 1082 students; Year 7 (n=533), Year 8 (n=377) and Year 9 (n=172). The prevalence of each group by year level is presented in this section. In this analysis, the New1 and EB groups are combined and labelled as EB_New1. Any student whose response pattern does not match with Expert, DLDN or EB_New is labelled as Unclassified. Figure 4 contains the prevalence of the four groups, based on their response patterns, by year level.

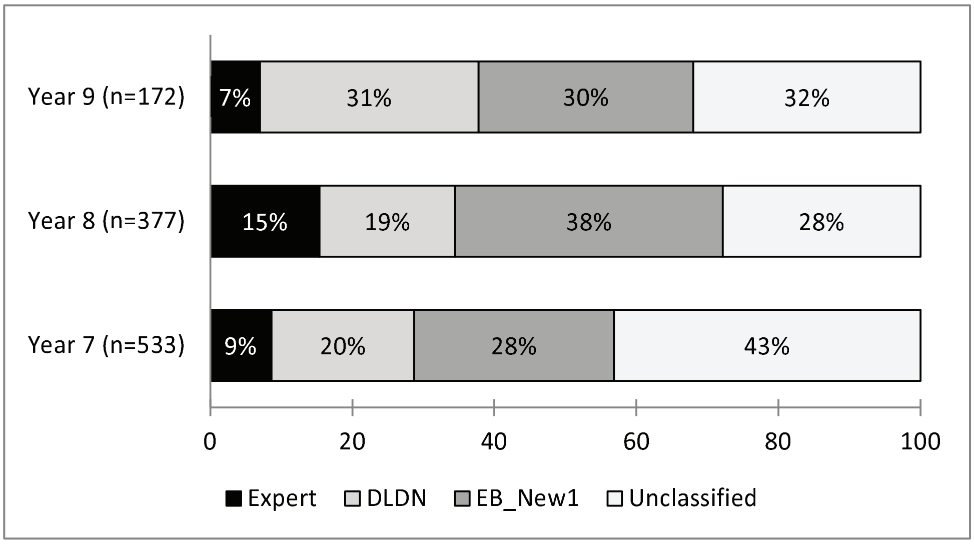

Figure 4. Prevalence of four groups based on response patterns, by year level.

The Year 7 data could be considered to be pre-teaching data; hence it is not a surprise that less than 10% of the Year 7 students answered as an Expert on the six tasks of interest. That this has increased to 15% in Year 8 is an improvement, but the low level in Year 9 is surprising. These are very similar to the results in Steinle et al. (2009) for students in Years 7 and 8, although there were no Year 9 students in that sample to compare with. This is also similar to the results noted by Fujii for the samples in Japan and the USA.

About 20% to 30% of each year level responded as DLDN and about 30% to 40% of each year level responded as EB_New1, which are similar to the results in Steinle et al. (2009). About 30% to 45% of each year level did not match any of the other response patterns and were labelled as Unclassified; this likewise compares with 30% to 50% of the previous study.

Conclusion

The focus of the study was to determine if the unexpected New1 response pattern found in Steinle et al. (2009) was essentially students who had the Empty Box misconception. We have support for this hypothesis; firstly, both groups responded in a similar, predicted way to a new task, and secondly, they responded differently to the two other groups of students.

Overall, there is considerable consistency between the results of this study and the previous studies. The small percentage of Year 8 and 9 students who answer both items correctly (referred to as Experts in this paper) is a concern, and requires more investigation. We intend to further develop this smart-test by including parallel items so that the diagnosis is based on more responses and hence is more robust. We should also add in extra tasks, within each item, where the Empty Box students would choose the option *wrong*; for example, within Item 2269, where the equation is $x + y = 16$, "Jill wrote

$x = 7$, $y = 11$". This would separate those who choose Right for each task (perhaps due to non-engagement with the tasks) from those with the Empty Box misconception.

The findings of this study are important for classroom teachers and teacher educators. Knowing that there are likely to be many students in each Year 7, 8 and 9 mathematics class with the two main misconceptions, Empty Box and Different Letter means Different Number, should highlight the need for teachers to make the rules of assigning numerical values to letters more explicit. In particular, (i) multiple occurrences of a letter within one problem, all have the same value, and (ii) it is possible that two different letters (such as x and y) may, in fact, have the same value. In the first case, students are unlikely to make sense of the process of transforming equations (for example, $x + x + x = 12$ can be simplified to $3x = 12$ and then $x = 4$) if they believe that the different x's in the original equation can be different numbers (such as 9, 2 and 1). In the second case, students have not appreciated that an equation such as $x + y = 16$ can be used to represent a situation where there are two variables and that these do not necessarily have different numerical values. For these students, it is almost as if they believe that a graph of $x + y = 16$ has a hole at the point (8,8).

A major aim of the smart-test system is to increase teachers' mathematical pedagogical content knowledge. We hypothesised that putting data on their own students' thinking into teachers' hands would make research results come alive for teachers, and hence build their capacity. As reported in Steinle & Stacey (2012), there is some promising data from a survey of teachers using the system. Of the 127 responses to a multiple choice question 'As a result of using this quiz have you learned something useful for you as a teacher?', over 90% indicated that "yes" they did learn something useful; 58 (46%) chose "Yes, very valuable learning".

As noted by Asquith et al. (2007), while the middle school teachers in their USA sample were aware that some students would not have a relational definition of the equals sign, the teachers overestimated their students' ability. The purpose of the smart-test system is so that any teacher can obtain data on their students to inform their future teaching.

We conclude with a quote from Steinle et al. (2009, p. 498):

> For approximately 90% of these Year 8 students, then, mathematics lessons containing algebra are rendered incomprehensible; these students are trying to learn procedures, without meaning, carried out on letters with the wrong meaning.

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